

ON VARIOUS GEOMETRIES GIVING A UNIFIED ELECTRIC AND GRAVITATIONAL THEORY

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1. Recently Einstein¹ has deduced a unified theory of electric and gravitational fields by forming the Euler equations for the integral of a certain scalar density. I show in the present paper that his new equations can be obtained by direct generalization of the equations of the gravitational field previously given by him. The process of generalization consists in abandoning assumptions of symmetry and in adopting a definition of covariant differentiation which is not the usual one, but which reduces to the usual one in case the connection is symmetric.

In the final section I show that the adoption of the ordinary definition of covariant differentiation leads to a geometry which includes as a special case that proposed by Weyl² as a basis for the electric theory; further, that the asymmetric connection for this special case is of the type adopted by Schouten³ for the geometry at the basis of his electric theory.

2. Einstein's equations for the simple gravitational field are

$$g_{ij,k} = 0, \tag{2.1}$$

$$R_{ij} = 0. \tag{2.2}$$

They involve a fundamental (metric) tensor g_{ij} and an affine connection Γ_{ij}^k , both of which are symmetric in i and j . The symbol $g_{ij,k}$ denotes the covariant derivative of g_{ij} ,

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{aj} \Gamma_{ik}^a - g_{ia} \Gamma_{jk}^a, \tag{2.3}$$

and R_{ij} denotes the Ricci tensor formed for the Γ 's.

Let us now consider an asymmetric connection H_{jk}^i and an asymmetric tensor h_{ij} , setting

$$H_{jk}^i = \Gamma_{jk}^i + \Omega_{jk}^i, \tag{2.4}$$

$$h_{ij} = g_{ij} + \omega_{ij}, \tag{2.5}$$

where Γ_{jk}^i and g_{ij} indicate the symmetric parts, and Ω_{jk}^i and ω_{ij} , the skew symmetric. If we now postulate that the general equations of the gravitational and electric field are those obtained by abandoning the assumptions of symmetry in (2.1) and (2.2), we get

$$h_{ij/k} = 0, \tag{2.6}$$

$$Z_{ij} = 0, \tag{2.7}$$

where

$$Z_{jkl}^i = \frac{\partial H_{jk}^i}{\partial x^l} - \frac{\partial H_{jl}^i}{\partial x^k} + H_{jk}^\alpha H_{al}^i - H_{jl}^\alpha H_{ak}^i,$$

and

$$Z_{ij} = Z_{ij\alpha}^\alpha.$$

are the natural generalizations of the curvature and Ricci tensors, respectively.

We have yet to define the symbol appearing in (2.6) by choosing a generalization of the covariant derivative appearing in (2.1). If we write

$$h_{ij/k} = \frac{\partial h_{ij}}{\partial x^k} - h_{aj} H_{ik}^\alpha - h_{ia} H_{kj}^\alpha, \quad (2.8)$$

then (2.6) and (2.7) are the equations which Einstein gets after assuming

$$\Omega_{\alpha k}^\alpha = 0 \quad (2.9)$$

in order to make his equations agree with experimental facts.⁴ In the present mode of deduction no such additional assumption is necessary. The geometrical significance of conditions (2.9) is indicated in the next section.

3. It can be shown⁵ that if we transform the connection H_{jk}^i by the formulas

$$H_{jk}^i = H_{jk}^{\prime i} + 2\delta_j^i v_k, \quad (3.1)$$

where v_k is an arbitrary vector, that infinitesimal parallelism (and consequently paths) will be preserved in the sense that the direction of the vector obtained by displacement of a given vector through dx is the same for the laws of displacement determined by the two sets of H 's.

Under a transformation such as (3.1) we have also

$$\Omega_{jk}^i = \Omega_{jk}^{\prime i} + \delta_j^i v_k - \delta_k^i v_j. \quad (3.2)$$

Hence it is seen that by choosing v_j properly we can realize conditions (2.9) with preservation of displaced directions.

Defining $h_{ij/k}$ by (2.8), we can write the Einstein equations corresponding to (2.6) as

$$h_{ij/k} = h_{ij} \varphi_k + h_{ik} \varphi_j, \\ \varphi_k = -\frac{2}{n-1} \Omega_{\alpha k}^\alpha. \quad (3.3)$$

These are, of course, his equations before assumption (2.9) is made. It is easily shown that equations (3.3) are invariant under the transformation (3.1), whereas the equations (2.7) are not.

As a consequence of (3.3), we have on taking $\varphi_i = 0$, and on using (2.4) and (2.5)

$$g_{ij,k} = \omega_{\sigma j} \Omega_{ik}^\sigma + \omega_{i\sigma} \Omega_{kj}^\sigma.$$

From these last equations it follows that

$$g_{ij,k} + g_{jk,i} + g_{ki,j} = 0. \tag{3.4}$$

Since the imposing of conditions (2.9) does not affect the geodesics, we can state⁶ the

THEOREM. *The equations of the paths (straightest lines) of Einstein's space have a homogeneous quadratic first integral whose coefficients are the components of the symmetric part of the fundamental tensor.*

4. The covariant derivative defined by (2.8) can be extended in a number of ways to tensors of any rank. One such way is the following. The term to be subtracted corresponding to a covariant index p is of the form $h_{i\dots\alpha\dots j} H_{pk}^\alpha$ or of the form $h_{i\dots\alpha\dots j} H_{kp}^\alpha$ according as p occupies an odd or even position among the subscripts of h counting from the left. It is readily shown that the resulting expressions are the components of a tensor.⁷

This definition, however, is not the one usually adopted.⁸ If applied to the product of two vectors $\varphi_i \psi_j$, the result is *not* $\varphi_i \psi_{j/k} + \varphi_{i/k} \psi_j$ as is the case with ordinary covariant differentiation.

5. We shall next derive a few properties of fundamental tensors satisfying

$$h_{ij;k} = 0, \tag{5.1}$$

where

$$h_{ij;k} = \frac{\partial h_{ij}}{\partial x^k} - h_{\alpha j} H_{ik}^\alpha - h_{i\alpha} H_{jk}^\alpha$$

is the ordinary covariant derivative of h_{ij} with respect to the connection H . Equations (5.1) are equivalent to the two sets $g_{ij;k} = 0$ and $\omega_{ij;k} = 0$, the first of which can be written as

$$g_{ij,k} = g_{\alpha j} \Omega_{ik}^\alpha + g_{i\alpha} \Omega_{jk}^\alpha, \tag{5.2}$$

where $g_{ij,k}$ is defined by (2.3).

From (5.2) it follows that equations (3.4) are fulfilled in this case also.⁹ The tensor g_{ij} , therefore, furnishes a quadratic first integral for the equations of the paths just as in the case of Einstein's space.

The conditions that such a tensor give a Weyl metric geometry for some affine connection with the same paths have been given previously.¹⁰ They are

$$2(n-1)g_{ij,k} + g^{\alpha\beta}(g_{jk} g_{\alpha\beta,i} + g_{ik} g_{\alpha\beta,j} - 2g_{ij} g_{\alpha\beta,k}) = 0. \tag{5.3}$$

From (5.2) we get

$$g^{\alpha\beta} g_{\alpha\beta,k} = 2\Omega_{\alpha k}^{\alpha}. \quad (5.4)$$

Substituting from (5.2) and (5.4) in (5.3) we find that equations (5.3) are here equivalent to

$$\Omega_{jk}^i - \frac{\delta_j^i}{n-1} \Omega_{\alpha k}^{\alpha} + \frac{\delta_k^i}{n-1} \Omega_{\alpha j}^{\alpha} = 0.$$

These conditions show that the connection is of the semi-symmetric type employed by Schouten in his electrical theory.¹¹

If in (3.1) we chose $v_k = \Omega_{\alpha k}^{\alpha}/(n-1)$, by means of (3.2) we see that Schouten's connection is rendered symmetric with preservation of displaced directions (parallelism).¹² This transformation changes equations (5.3) into

$$(g_{ij,k})' = 4g_{ij} \Omega_{\alpha k}^{\alpha}/(n-1),$$

the covariant derivative being formed with respect to Γ' . Written in the form $(g_{ij,k})' = -g_{ij} \psi_k$, these equations put in evidence the Weyl character of the geometry and show that the symmetric connection associated with Schouten's geometry is the affine connection of Weyl's.¹³

The geometries of Schouten and Weyl which enter here as particular cases of the geometry characterized by (5.1) are therefore equivalent. If we look upon (5.1), or (5.2), both as a means of rendering Schouten's geometry metric and of determining an asymmetric part for Weyl's symmetric affine connection, then we can say that the metric geometries of Schouten and Weyl are equivalent. It is to be noted, however, that an asymmetric part can be obtained from (5.2) for an arbitrary Weyl geometry, whereas the general Schouten geometry will not admit a fundamental tensor satisfying (5.2).

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¹ *Berliner Sitzungsberichte*, 1925, pp. 414-419.

² *Raum, Zeit, Materie*, 4th ed., p. 109, et seq.

³ *Proceedings, Amsterdam Academy*, Vol. 26, 1923, p. 850.

⁴ Einstein assumes (2.9) to make the right hand member of (3.3) vanish. He does not use the conditions in any other connection in the cited paper.

⁵ H. Friesecke, *Math. Annalen*, Vol. 94, 1925, p. 106.

⁶ Cf. Veblen and T. Y. Thomas, *Trans. Amer. Math. Soc.*, Vol. 25, 1923, p. 581, for the theorem (first proved for the Riemann geometry by Ricci and Levi-Civita) that (3.4) are the conditions for a quadratic first integral.

⁷ A simple method of proof, suggested by Professor Veblen, is the following. Subtract the expression defined above from the ordinary expression for the covariant derivative, which is known to be a tensor. The resulting difference is readily recognized as a tensor because of the tensor character of the skew symmetric part of the connection.

⁸ Cf. Schouten, *Ricci-Kalkül*, pp. 63-65.

⁹ I am indebted to Professor Veblen for this remark.

¹⁰ Cf. my paper "First Integrals in the Geometry of Paths," in these PROCEEDINGS, 12.

¹¹ Loc. cit.

¹² In a paper entitled "Asymmetric Displacement of a Vector" to appear in the *Transactions of the American Mathematical Society* I have established this and other properties of the semi-symmetric connection, together with properties of the general asymmetric connection.

¹³ Schouten in his *Ricci-Kalkül*, p. 223, remarks that his geometry of the semi-symmetric connection and that of Weyl have the same paths. He further states that it is impossible to transform his asymmetric connection into Weyl's symmetric connection by a transformation preserving paths (bahntreue Transformation) because an asymmetric connection remains asymmetric under such a transformation. The latter statement, however, is true only as a consequence of his definition of a "bahntreue Transformation" (loc. cit., p. 76) in which he makes the unnecessary assumption that the change of the skew symmetric part is zero. When one asks what change of connection preserves *all* displaced directions, one finds that the change affects both the symmetric and skew symmetric parts of the connection, and that the change in the symmetric part not only preserves the paths, but is the most general change that does. See my paper cited under note 12.

*STUDIES ON BIOCHEMICAL DIFFERENCES BETWEEN (+)
AND (-) SEXES IN MUCORS. 2. A PRELIMINARY RE-
PORT ON THE MANOILOV REACTION AND OTHER
TESTS¹*

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Since 1913,² attempts have been made in this laboratory to discover biochemical differences which it was believed must exist between the opposite sexes in the group of fungi known as *Mucors*. That chemical methods are capable of distinguishing races which appear identical morphologically has been shown in the case of the yellow-coned races of *Rudbeckia*.³ As outlined in our paper of last year,⁴ it has been our purpose to run through a series of easily applied tests on a series of representative sexual races before undertaking an intensive study of any single biochemical reaction. While this program was underway, our attention was called to the reaction of Manoilov⁵ for the identification of sexes which we have tested with a relatively large number of *Mucors* as well as of green plants. On account of a lack of any constant morphological difference in *Mucors*, which would enable one to determine which is male and which female, we have provisionally designated the opposite sexes by the signs (+) and (-). The *Mucors* reproduce chiefly by means of non-sexual spores. The various races, therefore, may be kept pure under cultivation without change of sex through vegetative reproduction.